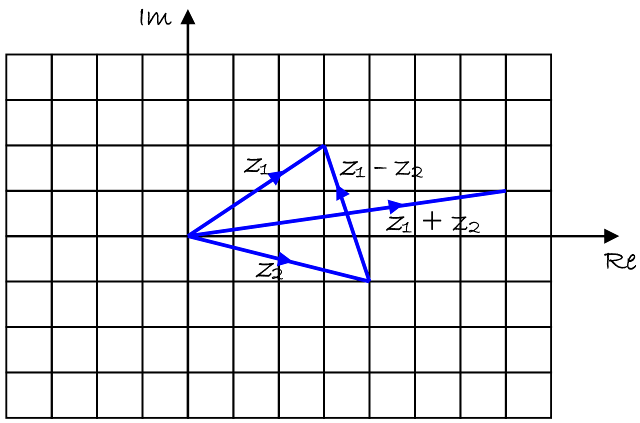
OCR Further Pure 1

Complex Numbers

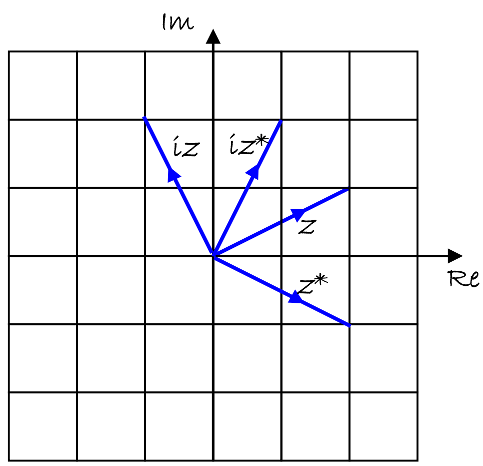
Section 2: Equations and geometrical representation

Solutions to Exercise

1.



2. $z = 2 + i$
$z^* = 2 - i$
$iz = 2i - 1 = -1 + 2i$
$iz^* = 2i + 1 = 1 + 2i$



(i) z^* is the reflection of z in the real axis
(ii) iz is a rotation of z through 90° anticlockwise.

3. $3 + i$ is a root, so $3 - i$ is also a root.
Therefore a quadratic factor is $(z-3-i)(z-3+i) = (z-3)^2 + 1$
$= z^2 - 6z + 10$
$1 + 3i$ is a root, so $1 - 3i$ is also a root.
Therefore a quadratic factor is $(z-1-3i)(z-1+3i) = (z-1)^2 + 9$
$= z^2 - 2z + 10$
So the equation is $(z^2 - 6z + 10)(z^2 - 2z + 10) = 0$
$z^4 - 8z^3 + 32z^2 - 80z + 100 = 0$

4. $1 - 2i$ is a root, so $1 + 2i$ is also a root.
So a quadratic factor is $(z - 1 + 2i)(z - 1 - 2i) = (z - 1)^2 + 4$
$= z^2 - 2z + 5$
$z^3 + z + 10 = (z^2 - 2z + 5)(z + 2)$
The real root is $z = -2$.

5. $(1 + i)^2 = 1 + 2i - 1 = 2i$
$(1 + i)^3 = 2i(1 + i) = 2i - 2 = -2 + 2i$
Substituting into $z^3 - 2z + k = 0$:
$-2 + 2i - 2(1 + i) + k = 0$
$-2 + 2i - 2 - 2i + k = 0$
$k = 4$

$1 + i$ is a root, so $1 - i$ is also a root.
So a quadratic factor is $(z - 1 - i)(z - 1 + i) = (z - 1)^2 + 1$
$= z^2 - 2z + 2$
$z^3 - 2z + 4 = (z^2 - 2z + 2)(z + 2)$
so the other two roots are $1 - i$ and -2.

6. $z = -1 + i$
$z^2 = (-1 + i)^2 = 1 - 2i - 1 = -2i$
$z^3 = -2i(-1 + i) = 2i + 2 = 2 + 2i$
$z^4 = (2 + 2i)(-1 + i) = -2 - 2 = -4$
Substituting into $z^4 - 2z^3 - z^2 + 2z + 10$:
$-4 - 2(2 + 2i) - (-2i) + 2(-1 + i) + 10$
$= -4 - 4 - 4i + 2i - 2 + 2i + 10$
$= 0$
so $-1 + i$ is a root.
Since $-1 + i$ is a root, $-1 - i$ is also a root
So a quadratic factor is $(z + 1 - i)(z + 1 + i) = (z + 1)^2 + 1$
$= z^2 + 2z + 2$
$z^4 - 2z^3 - z^2 + 2z + 10 = (z^2 + 2z + 2)(z^2 - 4z + 5)$
The other factors are the roots of the quadratic equation $z^2 - 4z + 5 = 0$

$$z = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 5}}{2}$$
$$= \frac{4 \pm \sqrt{-4}}{2}$$
$$= \frac{4 \pm 2i}{2}$$
$$= 2 \pm i$$

So the other roots are -1 – i, 2 + i and 2 – i.

7. $z = p + qi$

$z^2 = (p+qi)^2 = p^2 + 2pqi - q^2 = p^2 - q^2 + 2pqi$

$z^3 = (p^2 - q^2 + 2pqi)(p + qi)$

$= p(p^2 - q^2) + (2p^2q + q(p^2 - q^2))i - 2pq^2$

$= p^3 - 3pq^2 + (3p^2q - q^3)i$

Substituting into $z^3 + az + b = 0$:

$p^3 - 3pq^2 + (3p^2q - q^3)i + a(p + qi) + b = 0$

(i) Equating imaginary parts: $3p^2q - q^3 + aq = 0$

$3p^2 - q^2 = -a$

(ii) Equating real parts: $p^3 - 3pq^2 + ap + b = 0$

$p^3 - 3pq^2 + p(-3p^2 + q^2) + b = 0$

$-2p^3 - 2pq^2 = -b$

$2p(p^2 + q^2) = b$

(iii) From (i), $q^2 = 3p^2 + a$

Substituting into the result from (ii):

$2p(p^2 + 3p^2 + a) = b$

$2p(4p^2 + a) = b$

$8p^3 + 2ap - b = 0$

so p is a root of the equation $8x^3 + 2ax - b = 0$

8. $(a + bi)^2 = 3 - 4i$

$a^2 + 2abi - b^2 = 3 - 4i$

Equating imaginary parts: $2ab = -4 \quad \Rightarrow a = -\frac{2}{b}$

Equating real parts: $a^2 - b^2 = 3$

$$\frac{4}{b^2} - b^2 = 3$$

$$4 - b^4 = 3b^2$$

$$b^4 - 3b^2 - 4 = 0$$

$$(b^2 - 4)(b^2 + 1) = 0$$

$$b = \pm 2$$

$b = \pm 2 \Rightarrow a = \mp 1$

The square roots of 3 – 4i are 1 – 2i and -1 + 2i.