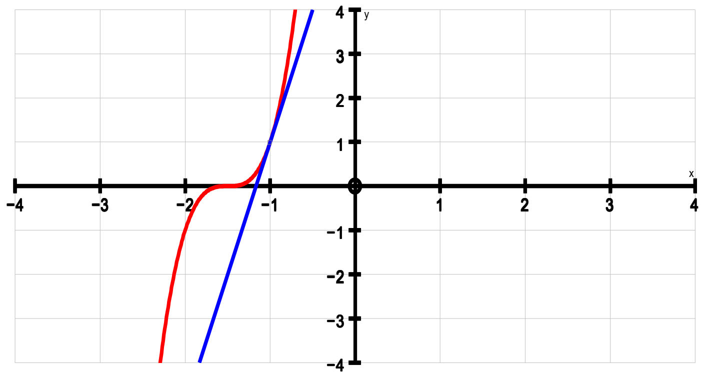
DIFFERENTIATION - the chain rule : A2 Core EX 4A

Suppose we wanted to consider the tangent to a curve such as $y=(2x+3)^3$ at the point $x = -1$.

The curve and the tangent….



But to differentiate the expression we would need to use the binomial expansion of $(a + b)^n$.

$$(2x+3)^3=(2x)^3+3(2x)^2(3)+3(2x)(3)^2+(3)^3$$

$$(2x+3)^3=8x^3+36x^2+54x+27$$

So the derivative $\dfrac{d}{dx}(2x+3)^3=24x^2+72x+54$

So at $x = -1$, the gradient of the tangent (see graph above) will be 6.

However ; We can use a simple substitution to avoid using the binomial method.

$y=(2x+3)^3$ Let $u=2x+3$

$y=u^3$ $\dfrac{du}{dx}=2$

$\dfrac{dy}{du}=3u^2$ => $\dfrac{dy}{dx}=3(2x+3)^2\times 2=6(2x+3)^2$

As before, at $x = -1$, the gradient is 6

Since derivatives behave like fractions, we use the idea that ;

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

Using $\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}$ Use this when the functions given are '*x* in terms of *y*'.

Example Find $\frac{dy}{dx}$ when $x = 4y^3 - \frac{1}{y^2}$

$$\frac{dx}{dy} = 12y^2 + \frac{1}{y^3}$$

The **reciprocal** of this derivative.... $\frac{dy}{dx} = \frac{1}{12y^2 + \left(\frac{1}{y^3}\right)} = \frac{y^3}{12y^5 + 1}$

RATES OF CHANGE (An application to the chain rule) : A2 Core Ex 4E

Consider a situation such as a spherical balloon being inflated. The surface area will increase as the radius of the balloon increases.

The rate at which the surface area increase $\frac{dS}{dt}$ is related to the rate at which the radius increases $\frac{dr}{dt}$

Suppose we are told the balloon's radius is increasing at 2cm s^{-1}.

Then $\frac{dr}{dt} = 2$

Question : Find the rate at which the ***surface area is increasing*** when r = 4cm.

Using the chain rule....... $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

So we also need $\frac{dS}{dr}$. This is simply the derivative of the relationship between V and r.

If $S = 4\pi r^2$ then $\frac{dS}{dr} = 8\pi r$

Therefore $\frac{dS}{dt} = 8\pi r \times 2 = 16\pi r$

When r = 4cm $\frac{dS}{dt} = 8\pi(4) \times 2 = 64\pi$ cm^2 s^{-1}

<u>**The surface area is increasing at a rate of 64π cm^2 s^{-1} when the radius is 4 cm.**</u>

GRADED HOMEWORK Exercise **Use the reverse of the sheet if necessary.**

1. Differentiate (a) $y = \sqrt{(3+2x)}$ (b) $y = \dfrac{1}{3x+4}$ (c) $y = \dfrac{1}{\sqrt{(1-2x)}}$

2. Differentiate these

(a) $y = \sqrt{x^2 + 3x}$ (b) $y = 4\left(1 - \dfrac{2}{x}\right)^3$

3. Find the value of $\dfrac{dy}{dx}$ at $x = 1$, for the curve $y = \dfrac{1}{(x^3+5)^2}$

4. Find the equation of the tangent to the curve $y = (2-x)^2$ at the point where $x = 3$, giving your answer in the form $y = mx + c$.

5. The radius of a circular disc is increasing at a constant rate of 0.003 cm s^{-1}. Find the rate at which the area is increasing when the radius is 20cm.

THE PRODUCT RULE : A2 Core Ex 4F

This is used when we want to differentiate functions that are **products**.

$$\text{If} \quad y = uv \quad \text{then} \quad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Examples Use the product rule to differentiate (i) $2x\ln x$ (ii) x^2e^x

THE QUOTIENT RULE : A2 CORE Ex 4F

Same idea as the product rule really. It is employed in order to differentiate **quotients** (fractions!)

$$\text{If} \quad y = \frac{u}{v} \quad \text{then} \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Obviously the quotient rule is a harder to use:

1. Be careful to get the numerator the right way round)
2. Remember to square '*v*' in the denominator.

Remember that the derivatives can represent the gradient value of tangents to these curves and may well need to be evaluated for given values of *x*.

Examples Use the quotient rule to differentiate the following

(i) $y = \frac{3x}{x^2+1}$ (ii) $y = \frac{(3-x)^2}{\sqrt{2x}}$