RATES OF CHANGE

The identity $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (chain rule) is useful for solving problems where three variables can be compared.

Suppose that the radius of a circle is increasing at the rate of 1mm per second.

In maths this is written as "The rate at which the radius is changing with respect to time" and is :

$$\frac{dr}{dt} = 1$$

Similarly, the rate at which the area is changing with respect to time is $\frac{dA}{dt}$ which will be different from $\frac{dr}{dt}$ of course.

We do not know the function that links A and t but we do know that $A = \pi r^2$.

So if we can find $\frac{dA}{dr}$ (just differentiate $A = \pi r^2$) and we know $\frac{dr}{dt} = 1$ then we can find $\frac{dA}{dt}$ by

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$

Example 1

A spherical balloon is being blown up so that its volume increases at a rate of 1.5cm^3/s

Think about the units here. Volume per second

Find the rate of increase of the radius when the volume of the balloon is 56cm^3.

We are looking for a relationship between *r* and *t* $\left(\frac{dr}{dt}\right)$

We know $\frac{dV}{dt} = 1.5$ and we also know that generally, $V = \frac{4}{3}\pi r^3$

So we can find $\frac{dV}{dr}$. $\frac{dV}{dr} = 4\pi r^2$

Just differentiate this

We need to use $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ but we need to rearrange to give

$\frac{dr}{dt} = \frac{dV}{dt} \div \frac{dV}{dr}$ = $\frac{dr}{dt} = 1.5 \div 4\pi r^2$

Remember that sometimes you need to change the subject of the chain rule.

Now substituting V = 56 into $V = \frac{4}{3}\pi r^3$ gives r = 2.373 (4sf)

So, $\frac{dr}{dt} = 1.5 \div 4\pi(2.373)^2 = 0.02120$

So the radius is increasing at a rate of 0.0212 cm/s

Example 2

An ink blot is increasing in area on blotting paper at the rate of 2.5 cm^2 per second. Find the rate at which the radius is changing when the area of the stain is 48cm^2.

Think about the units here. Area per

We know $\frac{dA}{dt} = 2.5$ and we also know generally that $A = \pi r^2$

Just differentiate

So we can find $\frac{dA}{dr} = 2\pi r$

Now we need $\frac{dr}{dt}$ so using the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

=> $\frac{dr}{dt} = \frac{dA}{dt} \div \frac{dA}{dr}$

We need to change the subject of the chain

$\frac{dr}{dt} = 2.5 \div 2\pi r$ **Substituting *A*= 48 into $A = \pi r^2$ gives r = 3.91**

(3sf)

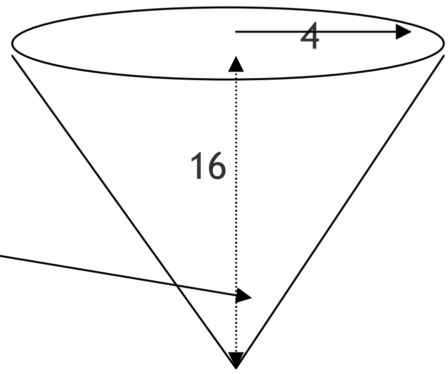
This is from the question

$$\frac{dr}{dt} = 2.5 \div 2\pi(3.91) = 0.102$$

<u>The radius is increasing at a rate of 0.102 cm per second</u>

Example 3 (quite difficult!)

A container in the shape of a cone has a height of 16 cm and the base radius of the cone is 4cm. The cone is inverted and is filled with liquid. If the liquid leaks out from the vertex at a rate of 4cm^3/s, find the rate of change of the depth of the liquid in the cone when half of the liquid has leaked out.



What do we know?

You need to consider

$V = \frac{1}{3}\pi r^2 h$ but $r = h$ tan14.036

$V = \frac{1}{3}\pi (h \tan 14)^2 h = \frac{1}{3}\pi (\tan 14)^2 h^3$

This is quite tough to differentiate. But remember, just numbers

$\frac{dV}{dt} = -4$ and $\frac{dV}{dh} = (\tan 14)^2 \pi h^2$ .....(31.664)

Volume is decreasing by 4 is shown by a negative

We need $\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$

The volume when full $= \frac{1}{3}\pi (4)^2 (16) = \frac{256\pi}{3}$ cm^3

So when half full = $\frac{128\pi}{3}$ cm^3 so **h =**

12.699..cm

$\frac{dh}{dt} = -4 \div 31.664$

h is found by rearranging the volume formula.

Decreasing at 0.126 cm/s