AQA Level 2 Further mathematics Number & algebra

Section 2: Algebraic manipulation

Notes and Examples

These notes contain subsections on

- **Factorising**
- **Factorising quadratic expressions**
- **The difference of two squares**
- **Changing the subject of a formula**
- **Algebraic fractions**
- **Completing the square**

Factorising

To factorise an expression, look for numbers and/or letters which are common factors of each term. We often talk about "taking out a factor" – this can cause confusion as it tends to make you think that subtraction is involved. In fact you are, of course, dividing each term by the common factor which you are "taking out".

You can check your answers by multiplying out the brackets.

Example 1
Factorise the following expressions.
(i) $6a + 12b + 3c$
(ii) $6x^2y - 10xy^2 + 2xy$
(iii) $3(x + y) - 2x(x + y)$

Solution
(i) 3 is a factor of each term.
$6a + 12b + 3c = 3(2a + 4b + c)$

(ii) $2xy$ is a factor of each term.
$6x^2y - 10xy^2 + 2xy = 2xy(3x - 5y + 1)$

(iii) $x + y$ is a factor of each term.
$3(x + y) - 2x(x + y) = (x + y)(3 - 2x)$

Factorising quadratic expressions

When you multiply out an expression involving two brackets, like $(2x-1)(x+3)$, you get a **quadratic expression**, which involves a term in x^2.

$$\begin{aligned}(2x-1)(x+3) &= 2x^2 - x + 6x - 3 \\ &= 2x^2 + 5x - 3\end{aligned}$$

Factorising a quadratic expression is the reverse of this process. Example 2 shows how you can do this.

Example 2

Factorise the expressions

(i) $x^2 + 4x + 3$

(ii) $x^2 - 4x - 12$

(iii) $2x^2 - 7x + 6$

Solution

(i) $x^2 + 4x + 3 = (x \ldots\ldots)(x \ldots\ldots)$

Start with an x in each bracket

$x^2 + 4x + 3 = (x + 1)(x + 3)$

You need two numbers whose sum is 4 and whose product is 3. These are +1 and +3.

(ii) $x^2 - 4x - 12 = (x \ldots\ldots)(x \ldots\ldots)$

Start with an x in each bracket

$x^2 - 4x - 12 = (x - 6)(x + 2)$

You need two numbers whose sum is –4 and whose product is –12. These are –6 and +2.

(iii) $2x^2 - 7x + 6 = (2x \ldots\ldots)(x \ldots\ldots)$

In this case you need to start with $2x$ in one bracket and x in the other.

$2x^2 - 7x + 6 = (2x - 3)(x - 2)$

It is not so straightforward to find the two numbers in this case, because of the $2x$ in one bracket. The two numbers must have a product of +6, and as the coefficient of x is negative, they must both be negative. Try the different possibilities (–1 and –6, or –2 and –3, in either order), until you find the correct one.

Expressions which involve terms in x^2, xy and y^2 can be factorised in a similar way – each bracket involves an x term and a y term.

Example 3

Factorise

(i) $x^2 + 2xy - 3y^2$

(ii) $2x^2 - 11xy + 12y^2$

Solution

(i) $x^2 + 2xy - 3y^2 = (x + 3y)(x - y)$

(ii) $2x^2 - 11xy + 12y^2 = (2x - 3y)(x - 4y)$

The difference of two squares

One important special case of a quadratic expression that you should recognise is called 'the difference of two squares'.

An example is the expression $x^2 - 4$. Both x^2 and 4 are squares, since 4 can be written as 2^2. This is a quadratic expression with no 'middle term'.

The expression $x^2 - 4$ can be factorised to give $(x + 2)(x - 2)$. Check this by multiplying out.

More generally, any expression of the form $a^2 - b^2$ can be written as $(a + b)(a - b)$. Look out for 'difference of two squares' expressions. The example below shows how using this result can save some work.

Example 4

Factorise fully $(3x+1)^2 - (2x-3)^2$.

Notice that this expression is the difference between two squares.

Solution

$$\begin{aligned}(3x+1)^2 - (2x-3)^2 &= \big((3x+1)+(2x-3)\big)\big((3x+1)-(2x-3)\big)\\ &= (3x+1+2x-3)(3x+1-2x+3)\\ &= (5x-2)(x+4)\end{aligned}$$

We can use $a^2 - b^2 = (a + b)(a - b)$, with a as $3x + 1$ and b as $2x - 3$.

In Example 4, you could multiply out both brackets, simplify and then factorise – this would involve a lot more work than the solution shown above!

You can see step-by-step examples of factorising quadratics in this ***PowerPoint presentation***.

You can also look at the ***Factorising quadratics video***.

For practice in examples like the ones above, try the interactive questions ***Factorising quadratics***.

You can also test yourself using the Flash resource ***Factorising quadratics***.

Changing the subject of a formula

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Changing the subject of a formula is similar to solving an equation, but you are working with letters rather than numbers.

Example 5

The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.
Make r the subject of this formula.

Solution

$$V = \pi r^2 h$$

Divide both sides by πh

$$\frac{V}{\pi h} = r^2$$

Square root both sides

$$\sqrt{\frac{V}{\pi h}} = r$$

Finish by writing the equation with r on the left side.

$$r = \sqrt{\frac{V}{\pi h}}$$

In the next example, two different methods are shown. The answers look a bit different but they are equivalent. Make sure that you can see how you could rewrite one solution to give the other.

Example 6

The surface area A of a cylinder with radius r and height h is given by $A = 2\pi r(h+r)$.
Make h the subject of this formula.

Solution (1)

$$A = 2\pi r(h+r)$$

Multiply out the brackets

$$A = 2\pi rh + 2\pi r^2$$

Subtract $2\pi r^2$ from each side

$$A - 2\pi r^2 = 2\pi rh$$

Divide each side by $2\pi r$

$$\frac{A - 2\pi r^2}{2\pi r} = h$$

Rewrite with h on the left

$$h = \frac{A - 2\pi r^2}{2\pi r}$$

Solution (2)

$$A = 2\pi r(h+r)$$

Divide each side by $2\pi r$

$$\frac{A}{2\pi r} = h + r$$

Subtract r from each side

$$\frac{A}{2\pi r} - r = h$$

Rewrite with h on the left side

$$h = \frac{A}{2\pi r} - r$$

In the next example, the new subject appears more than once. You need to collect the terms involving the new subject together and then factorise to isolate the new subject.

Example 7

Make x the subject of the formula $cx - a = a(b + x)$.

Solution

$cx - a = a(b + x)$ — Multiply out the brackets

$cx - a = ab + ax$ — Subtract ax from each side to collect the terms in x together

$cx - ax - a = ab$ — Add a to each side

$cx - ax = ab + a$ — Factorise

$(c - a)x = a(b + 1)$ — Divide both sides by $c - a$

$$x = \frac{a(b+1)}{c-a}$$

The ***Rearranging formulae video*** shows a number of examples of rearranging formulae.

Algebraic fractions

Algebraic fractions follow the same rules as numerical fractions. When adding or subtracting, you need to find the common denominator, which may be a number or an algebraic expression.

Example 8

Write as single fractions:

(i) $\dfrac{2x}{3} + \dfrac{x}{4} - \dfrac{5x}{6}$

(ii) $\dfrac{1}{2x} - \dfrac{1}{x^2}$

(iii) $\dfrac{2}{x-1} + \dfrac{3}{x+2}$

Solution

(i) The common denominator is 12, as 3, 4 and 6 are all factors of 12.

$$\frac{2x}{3}+\frac{x}{4}-\frac{5x}{6}=\frac{8x}{12}+\frac{3x}{12}-\frac{10x}{12}$$
$$=\frac{8x+3x-10x}{12}$$
$$=\frac{x}{12}$$

(ii) The common denominator is $2x^2$.

$$\frac{1}{2x}-\frac{1}{x^2}=\frac{x}{2x^2}-\frac{2}{2x^2}$$
$$=\frac{x-2}{2x^2}$$

(iii) The common denominator is $(x-1)(x+2)$.

$$\frac{2}{x-1}+\frac{3}{x+2}=\frac{2(x+2)}{(x-1)(x+2)}+\frac{3(x-1)}{(x+2)(x-1)}$$
$$=\frac{2x+4+3x-3}{(x-1)(x+2)}$$
$$=\frac{5x+1}{(x-1)(x+2)}$$

You are familiar with the idea of "cancelling" to simplify numerical fractions: for example, $\frac{9}{12}$ can be simplified to $\frac{3}{4}$ by dividing both the numerator and the denominator by 3. You can also cancel before carrying out a multiplication, to make the numbers simpler:

e.g. $\frac{\cancel{2}}{\cancel{3}}\times\frac{\overset{3}{\cancel{9}}}{\cancel{4}_{2}}=\frac{3}{2}$. The same technique can be used in algebra. As with factorising, remember that "cancelling" involves dividing, not subtracting.

Example 9

Simplify

(i) $\dfrac{6xy^3+2x^2y}{10x^2y}$

(ii) $\dfrac{3a}{a+1}\times\dfrac{2a+2}{a+2}$

It is very important to remember that you can only "cancel" if you can divide each term in both the numerator and denominator by the same expression. In this case, don't be tempted to divide by $2x^2y$ – although this is a factor of both $2x^2y$ and $10x^2y$, it is not a factor of $6xy^3$. In a case like this, it may be best to factorise the top first, so that it is easier to see the factors.

Solution

(i) $$\frac{6xy^3+2x^2y}{10x^2y}=\frac{2xy(3y^2+x)}{10x^2y}$$
$$=\frac{3y^2+x}{5x}$$

$2xy$ is a common factor of both top and bottom

(ii) Again, factorise where possible first.

$$\frac{3a}{a+1}\times\frac{2a+2}{a+2}=\frac{3a}{\cancel{a+1}}\times\frac{2\cancel{(a+1)}}{a+2}$$

$$=\frac{6a}{a+2}$$

$(a + 1)$ is a common factor of both top and

Notice that you cannot cancel a here, as it is not a factor of $a + 2$.

Sometimes algebraic expressions which look quite complicated can be simplified by factorising.

Example 10

Simplify $\dfrac{x^2-1}{x^2+2x-3}$.

It can be tempting to try to "cancel" x^2 from the top and the bottom. Don't! You can only cancel something which is a factor of the top and the bottom.

Solution

$$\frac{x^2-1}{x^2+2x-3}=\frac{(x+1)(x-1)}{(x+3)(x-1)}$$

$$=\frac{x+1}{x+3}$$

You can now cancel the factor $(x – 1)$ from the top and the bottom.

You may also find the Mathcentre video ***Simplifying algebraic fractions*** useful.

Completing the square

Sometimes it is useful to write a quadratic expression in the form $a(x + b)^2 + c$. This is called the completed square form. This form can be useful to help you find out more about a quadratic expression. For example, the expression $2(x – 1)^2 + 3$ has a minimum value of 3, since the squared term cannot be less than zero. This wouldn't be immediately obvious if the expression were given in its expanded form of $2x^2 – 4x + 5$.

The examples below demonstrate the technique of completing the square.

Example 11

Write the expression x^2+4x+7 in the form $(x + b)^2 + c$.

In this case you are asked for $(x + b)^2 + c$ rather than $a(x + b)^2 + c$. This is because the coefficient of x^2 is 1.

Solution

> First you need to find a quadratic expression which is a perfect square and which begins with $x^2 + 4x$.
> You do this by looking at the coefficient of x, in this case 4, and halving it. In this case you get 2.
> This tells you that the perfect square you need is $(x + 2)^2$.

$$(x+2)^2 = x^2 + 4x + 4$$
$$x^2 + 4x + 7 = x^2 + 4x + 4 + 3$$
$$= (x+2)^2 + 3$$

> The +3 'completes the square'

> This is why the technique is called 'completing the square'.

> $(x+2)^2$ is the 'square'

There are several different approaches to writing out the working. They are all basically the same, so if you have learnt a different way which suits you, then stick to it.

The next example shows a situation where the coefficient of x^2 is not 1.

Example 12

Write the expression $2x^2 – 6x + 1$ in the form $a(x + b)^2 + c$.

Solution

> Start by taking out the coefficient of x^2, in this case 2, as a factor.

$$2x^2 - 6x + 1 = 2(x^2 - 3x + \tfrac{1}{2})$$

> Now look at the expression inside the bracket. You need to find a quadratic expression which is a perfect square and starts with $x^2 - 3x$.
> Take the coefficient of x, which is –3, and halve it to get $-\frac{3}{2}$. The perfect square you need is therefore $\left(x - \frac{3}{2}\right)^2$.

$$(x - \tfrac{3}{2})^2 = x^2 - 3x + \tfrac{9}{4}$$

$$x^2 - 3x + \tfrac{1}{2} = x^2 - 3x + \tfrac{9}{4} - \tfrac{9}{4} + \tfrac{1}{2}$$
$$= (x - \tfrac{3}{2})^2 - \tfrac{9}{4} + \tfrac{1}{2}$$
$$= (x - \tfrac{3}{2})^2 - \tfrac{7}{4}$$
$$2x^2 - 6x + 1 = 2[(x - \tfrac{3}{2})^2 - \tfrac{7}{4}]$$
$$= 2(x - \tfrac{3}{2})^2 - \tfrac{7}{2}$$

In the next example, the coefficient of x^2 is negative. This can be dealt with by taking out a factor -1.

Example 14

Write the expression $5 + x - x^2$ in the form $p - q(x-r)^2$.

Solution

Start by taking out –1 as a factor.

$$5 + x - x^2 = -(x^2 - x - 5)$$

Now you need a quadratic expression which is a perfect square and starts with $x^2 - x$. Half the coefficient of x is $-\frac{1}{2}$, so the perfect square you need is $(x-\frac{1}{2})^2$.

$$(x-\tfrac{1}{2})^2 = x^2 - x + \tfrac{1}{4}$$

$$\begin{aligned} x^2 - x - 5 &= x^2 - x + \tfrac{1}{4} - \tfrac{1}{4} - 5 \\ &= (x-\tfrac{1}{2})^2 - \tfrac{1}{4} - 5 \\ &= (x-\tfrac{1}{2})^2 - \tfrac{21}{4} \end{aligned}$$

$$\begin{aligned} 5 + x - x^2 &= -[(x-\tfrac{1}{2})^2 - \tfrac{21}{4}] \\ &= \tfrac{21}{4} - (x-\tfrac{1}{2})^2 \end{aligned}$$

You may also find it helpful to look at the Flash resources on completing the square: ***Completing the square 1*** (in which the coefficient of x^2 is 1), ***Completing the square 2*** (in which the coefficient of x^2 is 2), and ***Completing the square 3*** (in which the coefficient of x^2 is -1). You may also find the Geogebra resource ***Completing the square*** useful – this uses area to show what is happening when you complete the square.

You can also look at the PowerPoint presentation ***Completing the square***, which is a step-by-step demonstration of some similar examples.